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Matrix Representation of the Stationary Measure for the Multispecies TASEP

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Abstract In this work we construct the stationary measure of the N species totally asymmetric simple exclusion process in a matrix product formulation. We make the connection between the matrix product formulation and the queueing theory picture of Ferrari and Martin. In particular, in the standard representation, the matrices act on the space of queue lengths. For $N > 2$ the matrices in fact become tensor products of elements of quadratic algebras. This enables us to give a purely algebraic proof of the stationary measure which we present for $N = 3$.

Keywords Totally asymmetric simple exclusion process · Multi-species systems · Stationary states · Matrix representation

1 Introduction

Models of diffusing particles with hard core interactions were first considered in the mathematical literature [1] and the name exclusion process was first coined by Spitzer [2]. In the totally asymmetric simple exclusion process (TASEP) particles jump only to the right

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on a one-dimensional lattice but cannot occupy the same site. Mathematical achievements include categorising the stationary measures for the process on $\mathbb{Z}$ and many results are summarised in the books by Liggett [3, 4].

Since the early 1990s the TASEP has been of considerable interest within the physics community as a prototypical model of nonequilibrium behaviour where, in the steady state, a current of particles is supported. In particular the model has been studied on the ring $\mathbb{Z}_L$ and also on a lattice of length L with open boundary conditions where particles enter at the left boundary and leave at the right boundary. Notable achievements have been the use of the Bethe ansatz to determine spectral properties of the transition rate matrix [5–8] and the determination of the stationary state in the open boundary case within a matrix product formulation [9].

A generalisation of the TASEP is to the case of several species of particle. In the two-species exclusion process containing first-class particles and second-class particles [12] both first and second-class particles hop to the right with rate 1. However if the site to the right of a first-class particle is occupied by a second-class particle the first and second-class particle exchange places with rate 1. Thus a second-class particle behaves as a hole from the point of view of a first-class particle but behaves as a particle from the point of view of a hole. The introduction of such a second class particle is a useful tool to study the microscopic structure of shocks [10–12]. Besides, the second-class particle problem arises naturally from coupling two TASEPs with different densities of particles [13]: the excess particles in the system with higher density acquire the dynamics of second-class particles. The stationary state of a system containing second and first-class particles has been obtained using the matrix product formulation by Derrida et al. [12]. Based on this work, Ferrari, Fontes and Kohayakawa [14] introduced a probabilistic construction of the measure. Angel [15] improved this construction providing a combinatorial description of the stationary state. In [16, 17], Ferrari and Martin showed that Angel's work could be interpreted as a queueing system and they generalized it to the N species case, for arbitrary N.

In the physics literature, the exclusion process with N species of particles was considered by Mallick, Mallick and Rajewsky [18] and studied for the case $N = 3$. This model, which we refer to as the N-TASEP, is defined by having site variables τ_i which may take values $0, 1, \ldots, N$ where N is the number of species. (Note that one could alternatively consider the state $\tau_i = 0$ (a hole) as a species which would imply a total of $N + 1$ species; we choose instead to use the more common convention.) The dynamics is defined as follows: each bond between neighbouring lattice sites has a bell which rings with rate 1. When the bell at bond $i, i + 1$ rings the site variables at i and $i + 1$ are exchanged provided $\tau_{i+1} = 0, \tau_i > 0$ or $\tau_{i+1} > \tau_i \geq 1$. This is equivalent to the following exchanges occurring with rate 1

$$K\ 0 \to 0\ K \quad \text{for } N \geq K \geq 1, \tag{1}$$

$$K\ J \to J\ K \quad \text{for } N \geq J > K \geq 1. \tag{2}$$

The construction of Ferrari and Martin [16, 17] couples N realizations of the TASEP in a special way, called the N-line process. To the N configurations in the N-line system one associates a configuration of the N-TASEP. Furthermore, each dynamical event of the N-line process corresponds precisely to a dynamical event in the N-TASEP. The steady state measure of the N-line system is just a uniform distribution of particles. This implies that one may sample the N-TASEP configurations with their stationary state probability by a two step procedure: (a) uniformly sampling a configuration of the N-line system of particles and (b) finding the associated configuration of the N-TASEP.

Our aim in this work is to invert this construction to obtain direct expressions for the steady state probabilities which generalise those already obtained for the two species case [12] and the three species case [18]. In doing so we shall see how the matrix product formulation generalises into a tensor product.

The paper is structured as follows. In Sect. 2 we review the known results on the stationary measure of the 2-TASEP and show how the matrix product representation [12] is related to the queueing representation [17]. In Sect. 3 we consider the N-TASEP and deduce a procedure for computing the stationary state probabilities. In Sect. 4 we construct a matrix product representation of the 3-TASEP stationary measure. In Sect. 5 we show how matrix product representations of the N-TASEP may be obtained recursively and we conclude in Sect. 6.

2 Two Species TASEP

In this section, we review the known solution of the two species TASEP. We also illustrate the equivalence between the matrix product solution of Derrida et al., the construction of Angel and the queueing process interpretation of Ferrari and Martin.

2.1 Construction of Angel that Generates the Stationary State

We begin by considering the construction of Angel for the two species TASEP on the ring $\mathbb{Z}_L$. (Note that we often use a different notation to [15] in order to avoid a clash with some standard notation from the matrix product formulation.) The construction is to consider a two-line configuration of particles (see Fig. 1). On line 1 there are P_1 particles distributed randomly (with at most one particle per site) and on line 2 there are $P_1 + P_2$ particles distributed randomly. Working from right to left we associate to each particle in line 1, the nearest particle, at the same site or to the left, in line 2 that has not already been associated to another particle. The associated particles in line 2 are then labelled 1 and the remaining P_2 unassociated particles are labelled 2. The empty sites of line 2 are labelled 0 thus each site of line 2 is labelled 0, 1 or 2. In this way a configuration of the two species TASEP containing P_1 first-class and P_2 second-class particles has been generated through the construction.

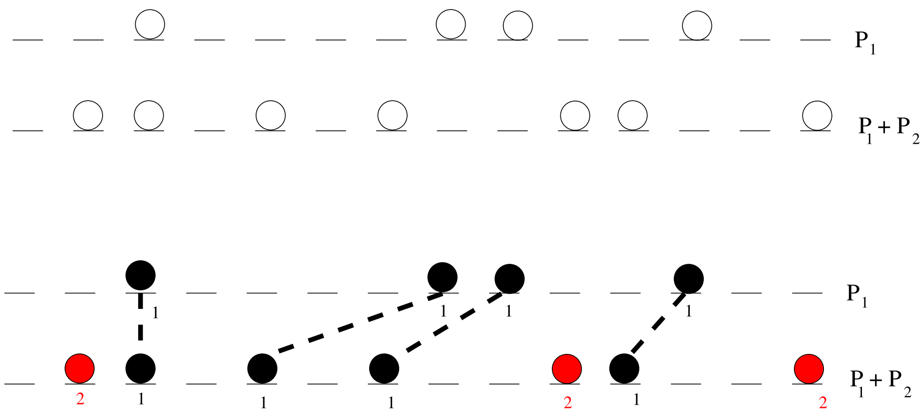

Fig. 1 Graphical representation of the construction for two species

The order used to choose the particles in line 1 to be associated to particles of line 2 (sequentially to the left, starting from the furthest site to the right in Fig. 1) does not affect the two species TASEP configuration that is generated. Indeed Angel gives the following non algorithmic definition of the labelling: (a) label 0 all the empty sites of line 2. (b) consider the sites i containing a particle of line 2; this particle is labelled 2 if and only if for all interval $[i, j] \subset \mathbb{Z}_L$ the number of particles in line 1 in $[i, j]$ is smaller than the number of particles in line 2 in $[i, j]$; otherwise the particle is labelled 1.

On the other hand, the particle in line 2 associated to a given particle in line 1 may depend on the order of choice and on the starting particle. For instance, keeping sequential order from right to left, if in Fig. 1 one starts with the third particle from the right in line 1, then it would be associated to the fourth particle from the right in line 2, while the second particle from the right in line 1 would be associated to the fifth particle from the right in line 2.

As noted in the introduction, uniformly sampling the 2-line configurations generates 2-TASEP configurations according to their stationary measure.

Angel showed that by uniformly sampling the two-line configurations, the configurations of the two-species TASEP, which we denote $\mathcal{C}$, are sampled with the following probabilities

$$\mathrm{P}(\mathcal{C}) = \frac{\prod_{j=1}^{P_2} \omega(B_j)}{Z(L, P_1, P_2)}. \tag{3}$$

Here $\omega(B_j)$ is the weight of the binary string B_j of 0,1 separating the second-class particles indexed by j and $j+1$ (here $j \in \mathbb{Z}_{P_2}$ indexes the second class particles). The normalization

$$Z(L, P_1, P_2) = \binom{L}{P_1}\binom{L}{P_1 + P_2} \tag{4}$$

just counts the number of possible 2-line configurations. Note that the form of the measure (3) implies a factorization of the stationary state about the positions of the second-class particles. The reason for the factorisation is, as can be seen from Fig. 1, that all 2-line configurations associated with a given 2-TASEP configuration must have the following properties: consider a site i, such that there is a particle labelled 2 at i in line 2, then i must be empty on line 1; moreover, no particle in line 1 to the right of i can be associated to a particle in line 2 to the left of i. This factorization property, which appeared in the matrix product formulation of Derrida et al. [12], was used in the construction of the stationary weights by [14].

The weights $\omega(B)$ are given by the following algorithm which we shall refer to as the *pushing procedure:* given the binary string B, one enumerates the number of strings which can be obtained from it by pushing the 1s to the right, in addition to the original string. For example from the string 110 one obtains 110, 101, 011. Thus $\omega(110) = 3$. Similarly, one can obtain from 1010 the strings 1010, 0110, 1001, 0101, 0011. Thus $\omega(1010) = 5$.

The measure given by (3) is stationary under the dynamics of the 2 species TASEP [12, 14, 15, 19]. Two key properties of this measure are (i) the factorisation of the probabilities of the 2-TASEP configurations about the position of the second class particles (ii) the weights $\omega(B)$ are given by the pushing procedure described above.

2.2 Matrix Product Solution of Derrida et al.

The matrix product formulation has been used to write down the stationary probabilities of various interacting particle models, thus allowing models to be solved through the explicit computation of physical quantities of interest such as currents, density profiles, correlation functions. It was first used to solve the TASEP on a lattice of length L with open boundary conditions [9]. It has been extended to the 2-TASEP on the ring $\mathbb{Z}_L$ [12], partially asymmetric processes and more general reaction-diffusion systems (for a review see [20]). In this matrix product formulation properties of the stationary measure manifest themselves in algebraic relations amongst the matrices involved. Some of these relations have been classified as quadratic algebras [21].

It is important to note that the measure (3), along with the calculation of the weights $\omega(B)$, is equivalent to that first obtained within the matrix product approach, as we now show. We recall that we use the variable $\tau_i = 0, 1, 2$ which implies that site i is empty, contains a first-class particle or contains a second-class particle, respectively. Let us denote by $\mathcal{C} = (\tau_1, \ldots, \tau_L)$, a configuration of the system. In the matrix product formulation [12] it has been proved that the stationary measure may be written as

$$\mathrm{P}(\mathcal{C}) = Z^{-1} W(\mathcal{C}), \tag{5}$$

where the weight of the configuration is given by

$$W(\mathcal{C}) = \mathrm{Tr}\left[\prod_{i=1}^{L} X_{\tau_i}\right] \tag{6}$$

and Tr means the trace of the product of matrices X_{τ_i}. The normalization $Z = Z(L, P_1, P_2)$ is chosen so that the sum of all the probabilities is equal to 1. The matrices X_{τ_i} are given by

$$X_0 = E, \qquad X_1 = D, \qquad X_2 = A, \tag{7}$$

that is: if the site is empty we write a matrix E; if the site contains a first-class particle we write a matrix D; if the site contains a second-class particle we write a matrix A.

The matrices D, E, A obey the algebraic rules

$$DE = D + E, \tag{8}$$

$$DA = A, \tag{9}$$

$$AE = A. \tag{10}$$

The only remaining condition to satisfy is that representations of E, D, A may be found which give well-defined values for the traces appearing in (5). This may be achieved as follows. Let $|n\rangle$ and $\langle n|$ be the column, respectively row, vector having a 1 in the nth coordinate and 0 in the other ones, $n = 0, 1, 2, \ldots$. Let A be the projector matrix

$$A = |0\rangle\langle 0| \tag{11}$$

then D, E may be chosen to be bidiagonal semi-infinite matrices

$$D = \sum_{n=0}^{\infty} |n\rangle\langle n| + |n\rangle\langle n+1|, \tag{12}$$

$$E = \sum_{n=0}^{\infty} |n\rangle\langle n| + |n+1\rangle\langle n|. \tag{13}$$

Writing out the matrices explicitly we have

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \ddots \\ 0 & 0 & 1 & 1 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \ddots \\ 0 & 1 & 1 & 0 & \ddots \\ 0 & 0 & 1 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{14}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{15}$$

Due to the form of A, (5) reduces to

$$\mathrm{P}(\{\tau_i\}) = Z^{-1} \prod_{j=1}^{P_2} \omega(B_j) \tag{16}$$

where B_j is as in (3), and $\omega(B_j)$ is now given by

$$\omega(B_j) = \langle 0| \prod_{i=1}^{l} X_{\tau_i} |0\rangle \tag{17}$$

where l is the length of the binary string B_j and i labels the entries in that string; X_{τ_i} is either a matrix D or a matrix E according to whether the entry τ_i in the string is 1 or 0.

The algebraic rules (9, 10) imply immediately that the weight of a string comprising a segment of consecutive zeros followed by a segment of consecutive ones is equal to 1. In other words, a string where any 0s are all to the left and any 1s are all to the right has weight 1:

$$\omega(0\cdots01\cdots1) = \langle 0|E\cdots ED\cdots D|0\rangle = \langle 0|0\rangle = 1. \tag{18}$$

Using rule (8) all binary strings can be reduced to strings of the above type and the weight of any string is easily computed. For example,

$$\begin{aligned} \omega(10) &= \langle 0|DE|0\rangle = \langle 0|D|0\rangle + \langle 0|E|0\rangle = 2, \\ \omega(110) &= \langle 0|DDE|0\rangle = \langle 0|DD|0\rangle + \langle 0|DE|0\rangle = 1+2=3, \\ \omega(1010) &= \langle 0|DEDE|0\rangle = \langle 0|DDE|0\rangle + \langle 0|EDE|0\rangle = 3 + \langle 0|DE|0\rangle = 5. \end{aligned} \tag{19}$$

This reduction procedure gives precisely the same result as the pushing procedure of Angel. In fact Lemma 2.3 of [14] proves that when ω is defined via the pushing procedure, the following relation holds

$$\omega(B10B') = \omega(B1B') + \omega(B0B') \tag{20}$$

for arbitrary finite binary sequences $B,\ B'$. But this is the same reduction formula that holds for the matrix representation:

$$\omega(B10B') = \langle 0|XDEX'|0\rangle \tag{21}$$

$$= \langle 0|XDX'|0\rangle + \langle 0|XEX'|0\rangle \tag{22}$$

where $X,\ X'$ are the matrix representation of the binary sequences $B,\ B'$, respectively. This shows that the definition of ω by the matrix formulation and that by the pushing procedure coincide.

The weights W of 2-TASEP configurations are computed from the weights ω of binary strings as follows. Recalling that N-TASEP configurations are translationally invariant under the periodic boundary conditions we have:

$$W(0210) = W(2100) = \omega(100) = 3, \tag{23}$$

$$W(0211021) = W(1021102) = \omega(10)\omega(110) = 6, \tag{24}$$

and the corresponding probabilities are given by

$$\mathrm{P}(0210) = \mathrm{P}(2100) = \frac{W(0210)}{Z(4,1,1)} = \frac{3}{24}, \tag{25}$$

$$\mathrm{P}(0211021) = \mathrm{P}(1021102) = \frac{W(1021102)}{Z(7,3,2)} = \frac{6}{735}, \tag{26}$$

where Z is defined in (4).

2.3 Queueing Interpretation of Ferrari and Martin

Ferrari and Martin used N-line configurations to generate N-TASEP configurations in terms of queueing processes. Here, we recall the interpretation in the case of the 2-species TASEP in terms of queueing processes and make the connection with the matrix product representation of the stationary measure.

We recall (see e.g. Fig. 2) that a 2-line configuration generates a 2-TASEP configuration. Since, as described above, the stationary state factorises about the positions of the second-class particles we only need to consider a binary string B of 1s and 0s between two 2s in a 2-TASEP configuration. There are several 2-line configurations which generate the string B. Those 2-line configurations must satisfy two conditions: (a) both lines 1 and 2 must contain the same number of particles, equal to the number of 1s in B and (b) line 2 coincides with B. The possibilities for line 1 are then generated from line 2 by pushing particles to the right. For example, the lower line of Fig. 2 has three strings of type B delimited by the three second class particles: $B_0 = 0$, $B_1 = 1010100$ and $B_2 = 100$. The upper-line string 1000011 is one of the strings producing B_1, the string 010 is one of the strings producing B_2 and the string 0 is the only string producing B_0.

Given a 2-line configuration one can associate to it the trajectory of the length of a queue. Consider the labels of the particles of line 2 as first or second class particles as given by Angel's algorithm (illustrated in Fig. 1). Time for the queue runs from right to left: at each site i it is assigned a time $t(i) = L - i$. The queue has length zero at the times corresponding to the positions of second class particles in line 2 (unused service times), a particle in line 1 represents an arrival time and a first class particle in line 2 represents a service time. At a given time $t(i)$ the length of the queue (constrained to be non-negative) increases by one

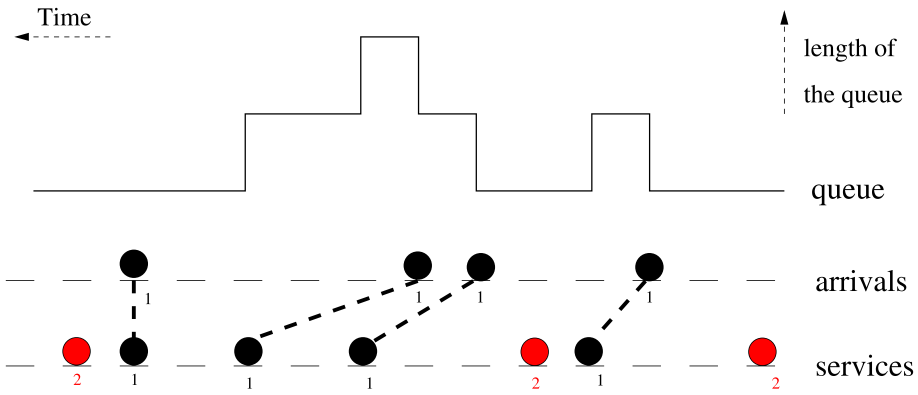

Fig. 2 Queueing picture for the two species case. Here time $t(i) = L - i$ increases from right to left, where i is site number

when a particle is present at site i in line 1 but not in line 2 (a new arrival occurs and is not serviced); the length of the queue decreases by one when a particle is present in line 2 but not in line 1 (a service occurs with no new arrival). If no particles are present in lines 1 and 2 (no service or new arrival occurs) or when particles are present in both lines 1 and 2 (a new arrival occurs and is serviced) the queue remains at the same length. The weight of a 2-TASEP string is then given by all possible queue trajectories, consistent with following constraints (i) the queue has length zero at the positions of the second class particles (ii) the effective service times of the queue are fixed by the positions of the first class particles. Since the full 2-line configuration can be retrieved by knowing the 2-TASEP configuration and the trajectory of the queue, to enumerate the ancestors of a 2-TASEP configuration it is enough to enumerate the queue trajectories compatible with it.

We now illustrate how the product of matrices A, D, E defined in (11, 12, 13) precisely enumerates the possible trajectories of the queue giving rise to a given 2-TASEP configuration. The right hand vector $|0\rangle$ represents an initial queue length of 0. At each service time of the queue we have a matrix D and at each non-service time a matrix E. A vector $|n\rangle$ represents the length of the queue. If the length of the queue just before a service time is $n > 0$ the action of D on $|n\rangle$ is

$$D|n\rangle = |n\rangle + |n-1\rangle. \tag{27}$$

The two terms represent the two possibilities at the service time: the first represents the service of a new arrival at that time, the second represents a service and no new arrival. If $n = 0$,

$$D|0\rangle = |0\rangle \tag{28}$$

which implies that a new arrival has to be serviced at this time, otherwise there would be an unused service which is forbidden.

Similarly, if the length of the queue is $n \geq 0$ just before a non-service time, the action of E on $|n\rangle$ is

$$E|n\rangle = |n\rangle + |n+1\rangle. \tag{29}$$

The first term represents no new arrival at that time, the second term represents a new arrival at that time.

The projector $|0\rangle\langle 0|$ at the left end of the string ensures that only trajectories of the queue which finish at length 0 are counted and the queue length is set to 0 for the start of the next string.

Remarks

(i) An alternative way to determine the queue length n at a given time $t(i) = L - i$ is the following. In Fig. 2 each particle in line 1 is associated to a particle in line 2 by a dashed line. The length of the queue at $t(i)$ is given by the number of dashed lines intersecting a vertical segment passing through i^-, i.e. just to the left of site i; vertical dashed lines do not affect the queue length.

(ii) The trajectory of the length of the queue in the queueing process with constraints described above is precisely a Motzkin path. This allows one to represent matrix product calculations in terms of ensembles of Motzkin paths see e.g. [22–24].

3 The N-Species TASEP

3.1 Construction for N Species of Particle

In this section we review how the construction for the 2-species case is extended to the N-species case [17]. For N species of particle we consider N-line configurations of particles. The first line comprises P_1 particles distributed randomly (with at most one particle per site). The second line comprises $P_1 + P_2$ particles distributed randomly and so on until the Nth line which comprises $P_1 + P_2 + \cdots + P_N$ particles distributed randomly. Initially, in this N-line configuration, particles are not differentiated into species. In the following we define the construction by which a species label is attributed to each of the particles. Once this has been done the Nth line is identified with an N-TASEP configuration.

We start with line 1 and associate each particle in line 1 to a particle in line 2 as in Sect. 2.1. This is done by beginning with a particle in line 1 and associating it with the nearest particle, at the same site or to the left, in line 2. We then take the next particle to the left in line 1 and associate it in the same way to the first unassociated particle in line 2. This process is continued until each of the particles in line 1 is associated with one particle in line 2. These P_1 particles in line 2 are then labelled 1 and the remaining P_2 unassociated particles in line 2 are labelled 2. The resulting labels do not depend on which particle we began with in line 1, as commented in Sect. 2.1.

We now proceed to associate the particles in line 2 with those in line 3. First we use the same procedure as described above to associate the particles labelled 1 in line 2 each to a particle in line 3. The P_1 associated particles in line 3 are then labelled 1. We then proceed to associate the particles labelled 2 in line 2 to P_2 of the unassociated particles in line 3 (ignoring the particles already labelled 1 in line 3). These particles in line 3 are then labelled 2 and the remaining P_3 unassociated particles in line 3 are labelled 3.

The procedure is then repeated up to line N and results in P_K of the particles in line N having label K where $1 \le K \le N$. The construction is illustrated by an example in the three species case in Fig. 3. Starting from the random distributions of particles in the N lines, one obtains a configuration of the N species TASEP with P_K particles of species K. The probability of the N-TASEP configuration so obtained is equal to its stationary probability under the N-TASEP dynamics. This was proven in [17]; in Sect. 4.4 we shall give an alternative proof, based on the matrix formulation, for the 3-TASEP.

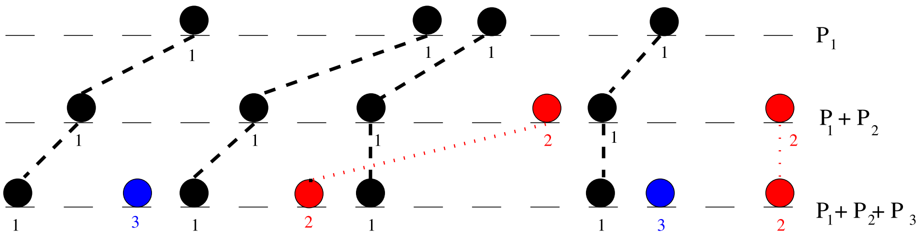

Fig. 3 Graphical representation of Ferrari-Martin's algorithm

Our aim is now to invert this construction. That is, for a given N-TASEP configuration, we wish to compute the probability that it is generated by the above procedure. This amounts to the combinatorial problem of counting the number of ways particles may be distributed in the N-line configuration such that the construction will lead to the desired N-TASEP configuration.

As has been discussed in Sect. 2.1, for the 2 species case Angel [15] gave such a construction and this is equivalent to the matrix product approach of Derrida et al. [12] (see Sect. 2.2). We showed this by using the queueing representation of Ferrari and Martin. In the following we first provide an algorithm, generalising the pushing procedure of Sect. 2 by which the probabilities can be computed. We then construct explicit matrices which compute the N-TASEP weights by book-keeping the generalized pushing procedure.

3.2 Ferrari and Martin's Construction and the Reverse Algorithm

As described above, in the N-species procedure of Ferrari and Martin a configuration of the N-TASEP is obtained from an N-line configuration, where each of the N lines consists of a single species TASEP configuration. The procedure may be viewed in the following way: from line 1 (a configuration of the single species TASEP) and line 2 one obtains a uniquely defined configuration of the 2-TASEP; from that configuration of the 2-TASEP and line 3 one constructs a configuration of the 3-TASEP and so on, until one reaches a configuration of the N-TASEP. Therefore, a given configuration of the N-TASEP arises from a whole set of $(N-1)$-TASEP configurations that we shall call its ancestors; each of these $(N-1)$-TASEP configurations arises in turn from a whole set of $(N-2)$-TASEP ancestors etc.

The stationary weight of the initial N-TASEP configuration is then given (but for an overall normalization constant) by the sum of the weights of the $(N-1)$-TASEP configurations that lead to it (its ancestors). Applying this procedure recursively, we observe that this stationary weight is given by the sum of the weights of its $(N-2)$-TASEP ancestor configurations. Finally, because the single species TASEP has a uniform steady state, the weight of any N-TASEP configuration is nothing but the total number of configurations of the single species TASEP from which it derives. Therefore, to calculate the weight of a given N-TASEP configuration we must determine the total number of 1-TASEP configurations that are its ancestors.

In the following we shall give a recursive algorithm to determine all the $(N-1)$-TASEP ancestors of a given N-TASEP configuration. Iterating this algorithm it is possible to obtain the total number of 1-TASEP ancestors of a given N-TASEP configuration; this number corresponds to the stationary weight of the N-TASEP configuration.

In order to simplify our discussion we shall first present this algorithm for the case of a 3-TASEP configuration, i.e. for a string of particles of classes 1, 2, 3 and holes (denoted by 0). We start with an initial 3-TASEP configuration:

(i) Freeze the positions of the 2s and the 3s. Construct all possible configurations obtained by pushing the 1s through the holes towards the right, until they hit a 2 or a 3 (i.e. a 1 can cross neither a 2 nor a 3).
$\rightarrow$ This procedure leads to many 3-TASEP configurations with various positions of the 1s. From now on the sites where the 1s are located will be passive.
(ii) Keep the positions of the 3s frozen and start moving the 2s. For each configuration obtained above, construct all possible configurations obtained by pushing the 2s through the holes towards the right, until they hit a 3. Note that the sites occupied by 1s are spectators and the 2s hop over them as if they do not exist.
$\rightarrow$ This procedure leads to many 3-TASEP configurations in which the positions of the 1s and the 2s are fixed.
(iii) Replace all the 3s by holes. We thus have obtained the complete set of 2-TASEP ancestors of the initial 3-TASEP configuration we started with.
(iv) The stationary weight of the initial 3-TASEP configuration (up to a global normalization constant) is the sum of the weights of its 2-TASEP ancestors.

Let us illustrate the algorithm with the explicit example of the string 2103

(i) By pushing 1s to the right we obtain the strings 2103, 2013.
(ii) By pushing 2s to the right (through the 1s), from 2103 we obtain 2103 and 0123 and from 2013 we obtain 2013 and 0213.
(iii) We replace 3s by 0s to obtain the strings 2100, 0120, 2010, 0210.
(iv) The weight of string 2103 in the 3-TASEP in terms of 2-TASEP weights is given by

$$W(2103) = W(2100) + W(0120) + W(2010) + W(0210),$$

which we may calculate, for example by using the matrix ansatz for the 2-TASEP (8–10) and (23), as $W(2103) = 3 + 1 + 2 + 3 = 9$.

It is easy to generalise to the N-species case and compute the weight of a N-TASEP configuration in terms of the weights of $(N-1)$-TASEP configurations

(i) Freeze the positions of the species $2, \ldots, N$. Construct all possible configurations obtained by pushing the 1s through the holes towards the right (a 1 cannot cross any species of particle).
(ii) Now in turn for $K = 2, \ldots, N-1$ push species K to the right keeping the positions of species $K+1, \ldots, N$ frozen and with species $1, \ldots, K-1$ spectators; i.e. for each configuration obtained from step 1, construct all possible configurations obtained by pushing the 2s through the holes to the right, allowing the 2s to hop over 1s; then push 3s to the right allowing 3s to hop over 2s and 1s, and so on until species $K-1$ have been pushed to the right, hopping over all other species except K.
(iii) In all of the N-TASEP configurations generated in step (ii), replace all the Ns by holes. We thus have obtained a whole set of $(N-1)$-TASEP configurations: this is the complete set of ancestors of the initial N-TASEP configuration we started with.
(iv) The sum of the stationary weight of all these $(N-1)$-TASEP ancestor configurations gives the stationary weight of the N-TASEP configuration.

We saw that ‘the pushing procedure’ of Angel is naturally implemented by the D, E and A matrices. The algorithm given above is also based on recursive pushing procedures and as we shall show in Sect. 4.2 can be encoded by a matrix ansatz; in this case the matrices for the N-TASEP are built by using the matrices for the $(N-1)$-TASEP as elements.

3.3 Queueing Interpretation of N-Species Construction

Ferrari and Martin also proposed a queueing interpretation for the multiline construction. The N lines of the N-species construction correspond to $N-1$ queues. The first line represents arrival times to the queue 1. The second line represents service times for queue 1. We continue using the convention that the queue time runs from right to left, so that the time $t(i)$ associated to site i is given by $t(i) = L - i$. As we have seen in Sect. 2.3, unused service times of queue 1 become second-class particles. Then when the particles of line 2 have been labelled either first or second-class, they represent the arrivals for queue 2. The arrivals are distinguished into first and second-class and the queue is a priority queue: at the service times (given by the particles in line 3) the highest priority waiting customer is always serviced first. That is, in queue 2 first-class arrivals are served before second-class arrivals. The output of these service times then become the arrival times for queue 3 with unused service times in queue 2 providing third-class arrivals to queue 3. This construction is iterated until the particles in line $N-1$, labelled $1, \dots, N-1$ provide the arrivals for queue $N-1$ and the particles in line N provide the service times for queue $N-1$. When the particles in line N are labelled $1, \dots, N$ they become the output of queue $N-1$, which corresponds to the N-TASEP configuration.

4 The Matrix Product Formulation

In this section, we show how the recursive construction for the N-TASEP, described in Sect. 3.2, can be encoded within the matrix product formulation, described in Sect. 2.2.

4.1 Definition of the Matrix Product Ansatz and Simple Examples

The matrix ansatz [9] provides a solution to the stationary master equation of the N-TASEP (made explicit later in (51)), as follows. First consider non-commuting matrices $X_0, X_1, \dots, X_N$, where X_K is associated to particles of class K (in particular X_0 is associated to holes, X_1 is associated to first-class particles etc.). The ansatz represents the stationary probability $\mathrm{P}(\mathcal{C})$ of a N-TASEP configuration $\mathcal{C} = (\tau_1, \dots, \tau_L)$ (where τ_i is equal to K if site i is occupied by a particle of class K) as a statistical weight $W(\mathcal{C})$ divided by a normalization Z

$$\mathrm{P}(\mathcal{C}) = \frac{1}{Z} W(\mathcal{C}) \tag{30}$$

where the weight is given by the trace of the product of L matrices, as follows

$$W(\mathcal{C}) = \mathrm{Tr}(X_{\tau_1} \dots X_{\tau_L}). \tag{31}$$

Here X_{τ_i} is equal to X_K if site i is occupied by a particle of class K ($K = 0, 1, \dots, N$) in configuration $\mathcal{C}$. The normalization factor Z (that depends on L and on all the P_K's where P_K represents the total number of particles of class K) ensures that $\sum_{\mathcal{C}} \mathrm{P}(\mathcal{C}) = 1$. We emphasize that the matrix formulation depends on the number of species. For example the matrices that represents first-class particles in the 2-TASEP and the 3-TASEP are not the same.

If the system contains only first-class particles and holes, it is well known that the stationary measure is uniform. Thus the particles and the holes may both be represented by one (a scalar) and the matrix ansatz reduces here to a trivial form.

For the 2-TASEP holes, first-class and second-class particles are represented respectively by the matrices $X_0 = E$, $X_1 = D$ and $X_2 = A$ (in the notation of [9, 12]) which satisfy (8, 9, 10). It is convenient to introduce matrices ϵ and δ defined by

$$E = \mathbf{1} + \epsilon, \qquad D = \mathbf{1} + \delta \tag{32}$$

where $\mathbf{1}$ is the identity matrix. Then, by (8, 9, 10), the matrices ϵ, A, δ generate the following quadratic algebra:

$$\begin{aligned} \delta\epsilon &= \mathbf{1}, \\ \delta A &= 0, \\ A\epsilon &= 0. \end{aligned} \tag{33}$$

4.2 Matrix Ansatz for 3-TASEP

We now present the matrix product formulation of the stationary state of the 3-TASEP.

$$X_1 = \mathbf{1} \otimes \mathbf{1} \otimes D + \delta \otimes \epsilon \otimes A + \delta \otimes \mathbf{1} \otimes E, \tag{34}$$

$$X_2 = A \otimes \mathbf{1} \otimes A + A \otimes \delta \otimes E, \tag{35}$$

$$X_3 = A \otimes A \otimes E, \tag{36}$$

$$X_0 = \mathbf{1} \otimes \mathbf{1} \otimes E + \mathbf{1} \otimes \epsilon \otimes A + \epsilon \otimes \mathbf{1} \otimes D. \tag{37}$$

Note that the matrices are generally sums of tensor products of three semi-infinite matrices used in the matrix product representation of the stationary state of the 2-TASEP. From the usual representation of the matrices A, δ and ϵ given in [9, 12] we obtain explicit expressions for the above matrices: they all have a block structure that is bidiagonal. Defining matrices F, G, H and R as

$$F = \begin{pmatrix} D & 0 & 0 & 0 & \dots \\ 0 & D & 0 & 0 & \ddots \\ 0 & 0 & D & 0 & \ddots \\ 0 & 0 & 0 & D & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}, \qquad G = \begin{pmatrix} E & 0 & 0 & 0 & \dots \\ A & E & 0 & 0 & \ddots \\ 0 & A & E & 0 & \ddots \\ 0 & 0 & A & E & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}, \tag{38}$$

$$H = \begin{pmatrix} A & E & 0 & 0 & \dots \\ 0 & A & E & 0 & \ddots \\ 0 & 0 & A & E & \ddots \\ 0 & 0 & 0 & A & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}, \qquad R = \begin{pmatrix} E & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix} \tag{39}$$

the matrix representation of (34–37) reads

$$X_1 = \begin{pmatrix} F & G & 0 & 0 & \dots \\ 0 & F & G & 0 & \ddots \\ 0 & 0 & F & G & \ddots \\ 0 & 0 & 0 & F & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}, \qquad X_2 = \begin{pmatrix} H & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}, \tag{40}$$

$$X_3 = \begin{pmatrix} R & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}, \qquad X_0 = \begin{pmatrix} G & 0 & 0 & 0 & \dots \\ F & G & 0 & 0 & \ddots \\ 0 & F & G & 0 & \ddots \\ 0 & 0 & F & G & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}. \tag{41}$$

All these matrices are triply infinite dimensional because their coefficients are themselves infinite dimensional matrices with elements D, A and E (which are also infinite dimensional matrices).

We define the trace of a product of X of matrices (34–37) as

$$\mathrm{Tr} X = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} [\langle l| \otimes \langle m| \otimes \langle n|]\, X\, [|l\rangle \otimes |m\rangle \otimes |n\rangle]\,. \tag{42}$$

Generally, using the definitions (34–37), we can write out any X as a sum of terms

$$X = \sum_i S_i \otimes U_i \otimes V_i. \tag{43}$$

where S_i, U_i and V_i are each single strings of fundamental matrices $\delta, \epsilon, \mathbf{1}, A$. A sufficient condition for the trace of X to be finite is that each string S_i, U_i and V_i contains at least one A. In turn, a sufficient condition for this is that X contains at least one X_3 and at least one X_2. In this case X necessarily contains a substring $X_2 \chi X_3$ where χ is a product of X_0 and X_1. Due to the form of (34, 37), the expansion (43) of χ can be written as

$$\chi = \sum_i s_i \otimes \epsilon^{n_i} \otimes v_i,$$

where $n_i \geq 0$ and v_i contains precisely n_i matrices A. Then we find that

$$\begin{aligned} X_2 \chi X_3 &= \sum_i \left[A s_i A \otimes \epsilon^{n_i} A \otimes A v_i E + A s_i A \otimes \delta \epsilon^{n_i} A \otimes E v_i E \right] \\ &= \sum_i \left[A s_i A \otimes \epsilon^{n_i} A \otimes A v_i E + A s_i A \otimes (1 - \delta_{n_i,0}) \epsilon^{n_i - 1} A \otimes E v_i E \right] \end{aligned} \tag{44}$$

where we have used the quadratic algebra (34) and $(1 - \delta_{n_i,0}) = 0$ if $n_i = 0$. Thus we see that (44) is a sum of strings of form (43) fulfilling the condition that each string S_i, U_i and

V_i contains at least one A. Thus, we have proven that if a string X of matrices X_{τ_i} contains at least one X_2 and at least one X_3, then TrX is finite.

It is important to note that we do not have a simple factorisation property as in the $N = 2$ case where A was a projector. Instead we find that

$$X_2 X_3 = A \otimes A \otimes A, \tag{45}$$

thus $X_2 X_3$ is a projector.

4.3 Matrices as Priority Queue Matrices

We now explain in the case $N = 3$ how these matrices may be obtained from the N-species queueing interpretation of the N-line configuration discussed in Sect. 3.3. In this case we have a 3-line configuration that represents two queues in tandem. Queue 1 has one type of customer which are considered to be first-class: Line 1 represents the arrival times of (first-class) customers in queue 1 and line 2 gives the service times of queue 1. The particles of line 2 are labelled 1 or 2 according to whether a service time is used or unused. Once labelled, the particles of line 2 become the arrival times to queue 2. Queue 2 is a priority queue containing first and second-class customers: any first-class customer is served before the second-class customers waiting in the queue. The particles of line 3 are the service times for queue 2. They are labelled by which class of customer is served; if a service is unused it is labelled 3.

We now consider the possible trajectories of the queue system. To do this we require 3 integer counters l, m, n: l is the number of first-class customers waiting in queue 2; m is the number of second-class customers waiting in queue 2; n is the number of first-class customers waiting in queue 1 (i.e. the length of queue 1). The three counters indicate the state of the system at each queue time $t(i) = L - i$ (which runs from right to left). The counters are therefore indexed by the times $t(i)$ associated to sites i, but we omit this in our notation.

Remark The values of the counters can be obtained directly from Fig. 3 as follows: for each site i, the counter l with index $t(i)$ represents the number of dashed lines crossing a vertical segment passing through i^- between lines 2 and 3; m represents the number of dotted lines crossing the same segment and n represents the number of dashed lines crossing the segment between lines 1 and 2; vertical dashed lines are not counted at all. The queue counters do not register (a) second class particles served at their arrival time, (b) first class particles served in both queues at their arrival time and (c) unused services in the third line. However, a 3-TASEP configuration and the trajectories of the three queues uniquely determine the 3-line configuration generating it. This implies that it is enough to enumerate the set of queues trajectories compatible with the 3-TASEP configuration we are computing the weight of.

The queue counters l, m, n can be represented by a state vector $|l\, m\, n\rangle$

$$|l\, m\, n\rangle \equiv |l\rangle \otimes |m\rangle \otimes |n\rangle \tag{46}$$

where $|l\rangle = 0$ for $l < 0$. We show now that the matrix product using X_0, X_1, X_2, X_3 defined in (34–37) precisely enumerates the possible trajectories of the state of the tandem queues giving rise to a given configuration $(\tau_1, \ldots, \tau_L)$.

Explicitly, the matrix element $\langle l'm'n'|X_{\tau_1} \ldots X_{\tau_1}|lmn\rangle$ is the number of queue histories, consistent with the configuration $\tau_1 \ldots \tau_k$, that begin with the queue counters taking values

l, m, n and end with the queue counters taking values l', m', n'. Thus the weight a configuration defined by the trace (42) of a product of matrices X_{τ_i} corresponds to the number of queue trajectories that return the queue counters after time $t(L)$ to any possible original setting l, m, n at $t(0)$, and that are consistent with the service times corresponding to the configuration $\tau_1 \dots \tau_L$.

We list the possible updates of the counters l, m, n at a given site (or time), according to the line 3 label of that site, i.e. the site variable τ_i in the N-TASEP configuration. Then from each possible update of the queue lengths we deduce the necessary action of the matrices X_i, $i = 0, 1, 2, 3$, on the state vector $|l\, m\, n\rangle$. Finally we can check from the definitions (34–37) of the actions of D, E, A (27, 28, 29) that $X_i|l\, m\, n\rangle$ produces the required update of the queue counters.

$\tau_i = 3$ In this case there is an unused service in queue 2 which implies $l = m = 0$. In queue 1 there may or may have not been an arrival therefore $n \to n$ or $n \to n + 1$. Thus, the action of X_3 must be

$$X_3|l\, m\, n\rangle = \delta_{l,0}\delta_{m,0}|0\rangle \otimes |0\rangle \otimes [|n\rangle + |n+1\rangle] = A \otimes A \otimes E|l\, m\, n\rangle \tag{47}$$

which recovers the matrix expression for X_3, (36).

$\tau_i = 2$ In this case a second-class service occurs in queue 2 which implies that the number of first-class customers $l = 0$ and there is no first-class arrival in queue 2. If there were a second-class arrival in queue 2 so that $m \to m$, it would imply $n = 0$ as there would have to be an unused service in queue 1. On the other hand, if there were no second-class arrival at queue 2 so that $m \to m - 1$ then there might or might not be a first-class arrival at queue 1 and $n \to n$ or $n \to n + 1$. Thus, the action of X_2 must be

$$\begin{aligned} X_2|l\, m\, n\rangle &= \delta_{l,0}|0\rangle \otimes \left[\, \delta_{n,0}|m\rangle \otimes |0\rangle + |m-1\rangle \otimes [|n\rangle + |n+1\rangle]\,\right] \\ &= [A \otimes \mathbf{1} \otimes A + A \otimes \delta \otimes E]\,|l\, m\, n\rangle \end{aligned} \tag{48}$$

which recovers the matrix expression for X_2, (35).

$\tau_i = 1$ In this case a first-class service occurs in queue 2. If there is also a first-class arrival at queue 2 then $l \to l$, $m \to m$ and $n \to n-1$ or n since there is a first-class service and possibly a first-class arrival at queue 1. If there is instead a second-class arrival at queue 2 (a second-class service in queue 1) then there must be no first-class customers in queue 1 and so $l \to l - 1$, $m \to m + 1$ and $n = 0$. Finally, if there is no arrival to queue 2 then there is no departure from queue 1 and there may or may not be an arrival at queue 1. Therefore $l \to l - 1$, $m \to m$ and $n \to n$ or $n + 1$.

Thus, the action of X_1 must be

$$\begin{aligned} X_1|l\, m\, n\rangle &= |l\rangle \otimes |m\rangle \otimes [|n\rangle + |n-1\rangle] + |l-1\rangle \otimes |m+1\rangle \otimes |0\rangle\delta_{n,0} \\ &\quad + |l-1\rangle \otimes |m\rangle \otimes [\,|n\rangle + |n+1\rangle\,] \\ &= |l\rangle \otimes |m\rangle \otimes D|n\rangle + \delta|l\rangle \otimes \epsilon|m\rangle \otimes A|n\rangle + \delta|l\rangle \otimes |m\rangle \otimes E|n\rangle \\ &= [\mathbf{1} \otimes \mathbf{1} \otimes D + \delta \otimes \epsilon \otimes A + \delta \otimes \mathbf{1} \otimes E]\,|l\, m\, n\rangle \end{aligned} \tag{49}$$

which recovers the matrix expression for X_1, (34).

$\tau_i = 0$ In this case there is no service at queue 2. If there is first-class arrival at queue 2 $l \to l + 1$, $m \to m$ and $n \to n - 1$ or n since there is a first-class service and possibly a first-class arrival at queue 1. If there is instead a second-class arrival at queue 2 (a second-class service in queue 1) then there must be no first-class customers in queue 1 and so $l \to l$,

$m \to m+1$ and $n=0$. Finally, if there is no arrival at queue 2 then there is no departure from queue 1 and there may or may not be an arrival at queue 1. Therefore $l \to l$, $m \to m$ and $n \to n$ or $n+1$. Thus, the action of X_1 must be

$$\begin{aligned} X_0|l\,m\,n\rangle &= |l+1\rangle \otimes |m\rangle \otimes [\,|n\rangle + |n-1\rangle\,] + |l\rangle \otimes |m+1\rangle \otimes |0\rangle \delta_{n,0} \\ &\quad + |l\rangle \otimes |m\rangle \otimes [\,|n\rangle + |n+1\rangle\,] \\ &= \epsilon|l\rangle \otimes |m\rangle \otimes D|n\rangle + |l\rangle \otimes \epsilon|m\rangle \otimes A|n\rangle + |l\rangle \otimes |m\rangle \otimes E|n\rangle \\ &= [\epsilon \otimes \mathbf{1} \otimes D + \mathbf{1} \otimes \epsilon \otimes A + \mathbf{1} \otimes \mathbf{1} \otimes E]\,|l\,m\,n\rangle \end{aligned} \tag{50}$$

which recovers the matrix expression for X_0, (37).

We now can interpret the result (45), that X_2X_3 is a projector, within the queueing representation. Since X_3 corresponds to an unused service in queue 2 we must have $l=m=0$ to begin with. Then, subsequent to the action of X_3, the action of X_2, corresponding to a second class service in queue 2, implies that there must have been a second class arrival in queue 2 to keep $m=0$. This in turn implies an unused service in queue 1 and $n=0$.

4.4 Algebraic Proof of the Matrix Product Ansatz

The matrix product ansatz may be proved independently of the queueing representation in an algebraic way. We shall use the technique of "hat matrices" to prove the ansatz (see e.g., [20, 25, 26] for more details).

We first recall the stationarity condition to be satisfied. The dynamics of the system can be encoded in a Markov matrix Q of size $\Omega \times \Omega$ where Ω is the total number of configurations of the system. The coefficient $Q(\mathcal{C}, \mathcal{C}')$ of this matrix represents the rate of transition from a configuration $\mathcal{C}'$ to a different configuration $\mathcal{C}$; $-Q(\mathcal{C}, \mathcal{C})$ is the total rate of exit from a given configuration $\mathcal{C}$. (Notice that this is the transpose of the usual generator matrix used in probability.) Thus the stationary probabilities must satisfy the stationary master equation

$$\sum_{\mathcal{C}'} Q(\mathcal{C}, \mathcal{C}')\mathrm{P}(\mathcal{C}') = 0. \tag{51}$$

Due to the local structure of the rules (1,2), Q can be written as a sum of local matrices that represent the transitions that take place at a bond $(i, i+1)$

$$Q = \sum_{i=1}^{L} Q_{i,i+1}. \tag{52}$$

where

$$Q_{i,i+1} = \mathbf{1}^{\otimes i-1} \otimes Q_{loc} \otimes \mathbf{1}^{\otimes L-i-1} \tag{53}$$

and Q_{loc} is a $(N+1)^2 \times (N+1)^2$ matrix whose off diagonal elements $Q_{loc}(\tau_i\tau_{i+1}; \tau_i'\tau_{i+1}')$ give the transition rate from local bond configuration $\tau_i'\tau_{i+1}'$ to $\tau_i\tau_{i+1}$ (for any i) and whose diagonal element $Q_{loc}(\tau_i\tau_{i+1}; \tau_i\tau_{i+1})$ gives minus the total transition rate out of local bond configuration $\tau_i\tau_{i+1}$.

Since the only transitions involved in the N-TASEP are exchanges at a bond, we have

$$\begin{aligned} &Q_{loc}(JK, KJ) = -Q_{loc}(KJ, JK) = 1, \quad \text{if } K \ge 1 \text{ and } J > K \text{ or } J = 0, \\ &Q_{loc}(K'J', KJ) = 0, \quad \text{otherwise,} \end{aligned}$$

where here K, J are indices that take values from 0 to N. When the steady state probabilities are written in the matrix product form (30) the matrix $Q_{i,i+1}$ (53) acts only on the ith and the $(i+1)$th matrices in the product through Q_{loc}. The stationarity condition (51) then may be written

$$\sum_{i=1}^{L} \mathrm{Tr}(X_{\tau_1} \dots X_{\tau_{i-1}} Y_{\tau_i,\tau_{i+1}} X_{\tau_{i+2}} \dots X_{\tau_L}) = 0 \tag{54}$$

where

$$Y_{\tau_i,\tau_{i+1}} = \sum_{\tau_i',\tau_{i+1}'} Q_{loc}(\tau_i \tau_{i+1}; \tau_i' \tau_{i+1}') X_{\tau_i'} X_{\tau_{i+1}'}. \tag{55}$$

That is,

$$\begin{aligned} &Y_{KJ} = -X_K X_J \quad \text{for all } K \geq 1 \text{ and } J > K \text{ or } J = 0, \\ &Y_{JK} = X_K X_J \quad \text{for all } K \geq 1 \text{ and } J > K \text{ or } J = 0, \\ &Y_{JJ} = 0 \quad \text{for all } J. \end{aligned} \tag{56}$$

The key point to prove the validity of the matrix ansatz is to show that $Y_{\tau_i,\tau_{i+1}}$ is a divergence-like term, i.e. there exist matrices $\hat{X}_\tau$ such that

$$Y_{\tau_i,\tau_{i+1}} = X_{\tau_i} \hat{X}_{\tau_{i+1}} - \hat{X}_{\tau_i} X_{\tau_{i+1}}. \tag{57}$$

Summation over i leads to a global cancellation in (54), proving thereby that the stationarity condition (51) is satisfied. Combining (56, 57), we obtain the conditions:

$$X_K X_J = \hat{X}_K X_J - X_K \hat{X}_J \quad \text{for all } K \geq 1 \text{ and } J > K \text{ or } J = 0, \tag{58}$$

$$X_K X_J = X_J \hat{X}_K - \hat{X}_J X_K \quad \text{for all } K \geq 1 \text{ and } J > K \text{ or } J = 0, \tag{59}$$

$$0 = X_J \hat{X}_J - \hat{X}_J X_J \quad \text{for all } J. \tag{60}$$

For $N = 2$, it turns out to be rather easy to solve the above equations (see e.g. [20]): indeed, one finds that (58–60) may be satisfied by choosing $\hat{X}_\tau$ to be scalars so that they *commute* with X_τ. Then (60) is immediately satisfied and (58, 59) reduce to 3 conditions

$$X_1 X_0 = \hat{X}_1 X_0 - \hat{X}_0 X_1, \tag{61}$$

$$X_1 X_2 = \hat{X}_2 X_1 - \hat{X}_1 X_2, \tag{62}$$

$$X_2 X_0 = \hat{X}_2 X_0 - \hat{X}_0 X_2. \tag{63}$$

Choosing $\hat{X}_1 = +1$, $\hat{X}_0 = -1$, $\hat{X}_2 = 0$ and $X_1 = D$, $X_0 = E$, $X_2 = A$ recovers (8–10).

For $N = 3$ it turns out that choosing $\hat{X}_\tau$ to be scalars does not allow (58–59) to be satisfied. Thus, the proof rests upon finding the 4 matrices $\hat{X}_K$ for $K = 0, 1, 2, 3$. We now write explicit forms for the hat matrices that fulfil the above relations when X_K are given by (34–37):

$$\begin{aligned} \hat{X}_1 &= (\mathbf{1} - \delta) \otimes \mathbf{1} \otimes \mathbf{1}, \\ \hat{X}_2 &= -A \otimes \delta \otimes \mathbf{1}, \\ \hat{X}_3 &= -A \otimes A \otimes \mathbf{1}, \\ \hat{X}_0 &= -X_0 + (\epsilon - \mathbf{1}) \otimes \mathbf{1} \otimes \mathbf{1}. \end{aligned} \tag{64}$$

It remains to verify that relations (58, 59, 60) are satisfied. Here we check a few relations involving X_1 and $\hat{X}_1$. For $J=1$ the rhs of (60) becomes

$$X_1\hat{X}_1 - \hat{X}_1 X_1 = (\mathbf{1}-\delta)\otimes\mathbf{1}\otimes D + \delta(\mathbf{1}-\delta)\otimes\epsilon\otimes A + \delta(\mathbf{1}-\delta)\otimes\mathbf{1}\otimes E$$
$$- (\mathbf{1}-\delta)\otimes\mathbf{1}\otimes D - (\mathbf{1}-\delta)\delta\otimes\epsilon\otimes A - (\mathbf{1}-\delta)\delta\otimes\mathbf{1}\otimes E = 0.$$

Thus (60) is satisfied in the case $J=1$.

Using relations (34) we find

$$X_1 X_2 = A\otimes\mathbf{1}\otimes DA + A\otimes\delta\otimes DE = A\otimes\mathbf{1}\otimes A + A\otimes\delta\otimes(D+E),$$

and

$$\begin{aligned}
\hat{X}_1 X_2 - X_1\hat{X}_2 &= A\otimes\mathbf{1}\otimes A + A\otimes\delta\otimes E - (-A\otimes\delta\otimes D)\\
&= X_1 X_2,\\
X_2\hat{X}_1 - \hat{X}_2 X_1 &= A(\mathbf{1}-\delta)\otimes\mathbf{1}\otimes A + A(\mathbf{1}-\delta)\otimes\delta\otimes E\\
&\quad - (-A\otimes\delta\otimes D - A\delta\otimes\delta\epsilon\otimes A - A\delta\otimes\delta\otimes E),\\
&= X_1 X_2
\end{aligned}$$

thus (58, 59) are satisfied for the case $K=1$, $J=2$. Similarly, all relations (58, 59, 60) may be verified.

4.5 Comparison to the Solution of Mallick, Mallick and Rajewsky

Mallick, Mallick and Rajewsky [18] found the 3-species stationary measure using a matrix Ansatz, we call their solution MMR matrices. The main differences with our approach are the following. (i) MMR matrices are doubly infinite while ours are triply infinite. (ii) MMR matrices may have negative entries (or even imaginary entries in an equivalent representation) and in this case no queueing interpretation is available. (iii) The trace of MMR matrices is finite (and real and positive) only when there is a matrix A_3 corresponding to the 3rd class particle at the right end of the string, i.e. the matrices cannot be multiplied in any order. For our matrices, however, the trace of $X_{\tau_1}\dots X_{\tau_L}$ is the same as the trace of $X_{\tau_2}\dots X_{\tau_L}X_{\tau_1}$, etc. (iv) The proof of the finiteness of the trace of the MMR matrices is rather involved, while our proof at the end of Sect. 4.2 is elementary. (v) So far MMR matrices could not be generalized to the N-class TASEP for $N>3$, while using the queueing representation in the next section we show how our matrices generalize to $N>3$.

5 Hierarchical Matrix Ansatz for the Multispecies ASEP

In this section, we generalize the previous construction to the multispecies totally asymmetric exclusion process on the ring $\mathbb{Z}_L$ with N classes of particles for any $N>1$. We show that a matrix ansatz for a system containing N classes of particles (plus holes) can be constructed recursively knowing a matrix ansatz for a system with $N-1$ classes of particles (plus holes). We shall simply present the results here and give some examples; algebraic proofs and generalizations can be found in [29].

The matrices $X_K^{(N)}$ at level N are obtained by making tensor products of the $X_M^{(N-1)}$ defined at level $N-1$ with some matrices a_{KM} constructed from ϵ, A, δ and $\mathbf{1}$. The matrix ansatz is given by

$$X_0^{(N)} = \sum_{M=0}^{N-1} a_{0M}^{(N)} \otimes X_M^{(N-1)}, \tag{65}$$

$$X_K^{(N)} = a_{K0}^{(N)} \otimes X_0^{(N-1)} + \sum_{M=K}^{N-1} a_{KM}^{(N)} \otimes X_M^{(N-1)} \quad \text{for } 1 \le K \le N. \tag{66}$$

We emphasize that in this section our notation for the matrix $X_K^{(N)}$ has two indices: the lower index K denotes the class of the particle represented by the matrix, whereas the upper index N gives the total number of classes considered in the system.

The fundamental building blocks to construct the $a_{KM}^{(N)}$ matrices are the matrices ϵ, A, δ and $\mathbf{1}$ (identity). The $a_{KM}^{(N)}$ are then given by

$$a_{00}^{(N)} = \mathbf{1}^{\otimes(N-1)}, \tag{67}$$

$$a_{0M}^{(N)} = \mathbf{1}^{\otimes(M-1)} \otimes \epsilon \otimes \mathbf{1}^{\otimes(N-M-1)} \quad \text{for } 1 \le M \le N-1. \tag{68}$$

For $K \ge 1$ we have

$$a_{K0}^{(N)} = A^{\otimes(K-1)} \otimes \delta \otimes \mathbf{1}^{\otimes(N-K-1)} \quad \text{for } 1 \le K \le N-1, \tag{69}$$

$$a_{KK}^{(N)} = A^{\otimes(K-1)} \otimes \mathbf{1}^{\otimes(N-K)}, \tag{70}$$

$$a_{KM}^{(N)} = A^{\otimes(K-1)} \otimes \delta \otimes \mathbf{1}^{\otimes(M-K-1)} \otimes \epsilon \otimes \mathbf{1}^{\otimes(N-M-1)} \quad \text{for } 1 \le K < M \le N-1, \tag{71}$$

$$a_{N0}^{(N)} = A^{\otimes(N-1)}. \tag{72}$$

It is understood in the formulae above that any matrix raised to a tensor-power equal to zero is equal to the scalar 1 which can be removed from the tensor product.

Note from (65, 66) that the matrices $X_K^{(N)}$ at level N are composed of tensor products of $\binom{N}{2}$ fundamental matrices ϵ, A, δ or $\mathbf{1}$.

5.1 Some Examples

Using the hierarchical matrix ansatz given above, we study explicitly the cases $N \le 3$.

For $N = 0$, the system does not contain any particles but only holes. There is only one configuration which has probability 1. Thus, we *define* $X_0^{(0)} = 1$.

For $N = 1$, we obtain from equations (65) and (66), using the fact that $X_0^{(0)} = 1$

$$X_0^{(1)} = a_{00}^{(1)} \quad \text{and} \quad X_1^{(1)} = a_{10}^{(1)}. \tag{73}$$

But from (67) and (72) we find that $a_{00}^{(1)} = a_{10}^{(1)} = 1$ and we recover the fact that for a system with only one class of particles the stationary measure is uniform and therefore the matrix ansatz is trivial.

For $N = 2$, we find from equations (67), (68), (70) and (72), that $a_{00}^{(2)} = \mathbf{1}$, $a_{01}^{(2)} = \epsilon$, $a_{10}^{(2)} = \delta$, $a_{11}^{(2)} = \mathbf{1}$, and $a_{20}^{(2)} = A$. Then, from the recursion relations (65) and (66), and using the fact that $X_0^{(1)} = X_1^{(1)} = 1$, we deduce that

$$X_0^{(2)} = a_{00}^{(2)} + a_{01}^{(2)} = \mathbf{1} + \epsilon = E, \qquad X_1^{(2)} = a_{10}^{(2)} + a_{11}^{(2)} = \mathbf{1} + \delta = D, \quad \text{and} \tag{74}$$

$$X_2^{(2)} = a_{20}^{(2)} = A. \tag{75}$$

We retrieve the fundamental matrix ansatz (32).

For $N = 3$, using the recursion relations (65) and (66), the definitions (67–72) and the results (74), (75), we obtain

$$\begin{aligned} X_0^{(3)} &= a_{00}^{(3)} \otimes X_0^{(2)} + a_{01}^{(3)} \otimes X_1^{(2)} + a_{02}^{(3)} \otimes X_2^{(2)} \\ &= \mathbf{1} \otimes \mathbf{1} \otimes E + \epsilon \otimes \mathbf{1} \otimes D + \mathbf{1} \otimes \epsilon \otimes A, \end{aligned} \tag{76}$$

$$\begin{aligned} X_1^{(3)} &= a_{10}^{(3)} \otimes X_0^{(2)} + a_{11}^{(3)} \otimes X_1^{(2)} + a_{12}^{(3)} \otimes X_2^{(2)} \\ &= \delta \otimes \mathbf{1} \otimes E + \mathbf{1} \otimes \mathbf{1} \otimes D + \delta \otimes \epsilon \otimes A, \end{aligned} \tag{77}$$

$$X_2^{(3)} = a_{20}^{(3)} \otimes X_0^{(2)} + a_{22}^{(3)} \otimes X_2^{(2)} = A \otimes \delta \otimes E + A \otimes \mathbf{1} \otimes A, \tag{78}$$

$$X_3^{(3)} = a_{30}^{(3)} \otimes X_0^{(2)} = A \otimes A \otimes E, \tag{79}$$

and we retrieve the expressions (34–37).

6 Discussion

In this work we have considered the multispecies totally asymmetric exclusion process on the ring $\mathbb{Z}_L$. We have shown how the stationary measure may be written in a matrix product formulation, thus providing an algebraic proof of the stationary measure which we presented for the three species case $N = 3$. For arbitrary N we have shown how the matrix product formulation may be constructed in a hierarchical fashion, see details in [29].

Ferrari and Martin have constructed the stationary state of the N-TASEP as the output of N queues in series with N priority-classes of customers, for all N. The construction takes a N-line binary configuration sampled at random and produces a N-TASEP configuration whose resulting law is invariant for the N-TASEP. We have shown that the matrix ansatz for $N = 2$ of Derrida et al. [12] gives a mechanism to count the number of 2-line configurations producing a given 2-TASEP configuration; the matrices may be thought of as acting on the space of queue counters. For $N = 2$ there is just a single queue with one type of customer and the queue counter is simply the length of the queue. We have also extended the matrix ansatz for $N > 2$. In this case there are multiple priority queues and there are several queue counters representing the number of each class of customer in each queue. This results in the queue matrices acting on tensor product spaces and accordingly the 'matrices' of the matrix formulation become higher rank tensors. This relation between the matrices and queueing processes also provides us with a natural representation of the space on which the matrices act. Until now, it was believed that the matrices act on a purely formal 'auxiliary' space which did not have any physical interpretation.

The algebraic proof of the stationary measure for $N > 3$ relies on the existence of 'hat' matrices [20, 25, 26] described in Sect. 4.4. This is in contrast to the $N = 2$ case where the hat matrices were simply scalars and the relations obeyed by the matrices become a quadratic algebra, as in the $N = 1$ open-boundaries case of [9]. For $N > 2$ the relations between the matrices have a more complicated algebraic structure and it would be of interest to explore this further.

One advantage of the matrix product formulation of the stationary measure is that it provides a framework within which the calculation of quantities of physical interest, such as correlation functions, can be carried out. So far we have not attempted such calculations but it would be important to do so.

Finally, we mention that the multispecies TASEP may be generalised in several ways by introducing rates which differ from one or additional processes. For example, for $N = 1$ allowing particles to carry out forward exchanges with holes with rate p and backward exchanges with rate q generates the partially asymmetric exclusion process for which a matrix product formulation of the steady state on the open boundary system has been fully worked out [27, 28]. In the case $N = 2$ there are partially asymmetric generalisations which admit matrix product stationary states [12, 20]. So far, in [29], we have found a partially asymmetric generalization of the matrix product ansatz presented in Sect. 5. It would be of interest to understand how the queueing interpretation of the steady state should be modified.

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